

THE MAPPING i_2 ON THE FREE PARATOPOLOGICAL GROUPS

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ABSTRACT. Let $FP(X)$ be the free paratopological group over a topological space X . For each non-negative integer $n \in \mathbb{N}$, denote by $FP_n(X)$ the subset of $FP(X)$ consisting of all words of reduced length at most n , and i_n by the natural mapping from $(X \oplus X^{-1} \oplus \{e\})^n$ to $FP_n(X)$. In this paper, we mainly improve some results of A.S. Elford and P. Nickolas's [On the topology of free paratopological groups. II, *Topology Appl.*, 160(2013), 220–229.]. The main result is that the natural mapping $i_2 : (X \oplus X_d^{-1} \oplus \{e\})^2 \rightarrow FP_2(X)$ is a closed mapping if and only if every neighborhood U of the diagonal Δ_1 in $X_d \times X$ is a member of the finest quasi-uniformity on X , where X is a T_1 -space and X_d denotes X when equipped with the discrete topology in place of its given topology.

1. INTRODUCTION

In 1941, free topological groups were introduced by A.A. Markov in [9] with the clear idea of extending the well-known construction of a free group from group theory to topological groups. Now, free topological groups have become a powerful tool of study in the theory of topological groups and serve as a source of various examples and as an instrument for proving new theorems, see [1].

As in free topological groups, S. Romaguera, M. Sanchis and M.G. Tkachenko in [12] defined free paratopological groups and proved the existence of the free paratopological group $FP(X)$ for every topological space X . Recently, A.S. Elford, F.C. Lin, P. Nickolas and N.M. Pyrch have investigated some properties of free paratopological groups, see [2, 3, 7, 8, 10, 11].

For each non-negative integer $n \in \mathbb{N}$, denote by $FP_n(X)$ the subset of $FP(X)$ consisting of all words of reduced length at most n , and i_n by the natural mapping from $(X \oplus X^{-1} \oplus \{e\})^n$ to $FP_n(X)$. In this paper, we mainly improve some results of A.S. Elford and P. Nickolas's. The main result is that the natural mapping $i_2 : (X \oplus X_d^{-1} \oplus \{e\})^2 \rightarrow FP_2(X)$ is a closed mapping if and only if every neighborhood U of the diagonal Δ_1 in $X_d \times X$ is a member of the finest quasi-uniformity on X , where X is a T_1 -space and X_d denotes X when equipped with the discrete topology in place of its given topology.

2. PRELIMINARIES

All mappings are continuous. We denote by \mathbb{N} and \mathbb{Z} the sets of all natural numbers and the integers, respectively. The letter e denotes the neutral element of a group. Readers may consult [1, 4, 6, 5] for notations and terminology not explicitly given here.

Recall that a *topological group* G is a group G with a (Hausdorff) topology such that the product mapping of $G \times G$ into G is jointly continuous and the inverse mapping of G onto itself associating x^{-1} with an arbitrary $x \in G$ is continuous. A *paratopological group* G is a group G with a topology such that the product mapping of $G \times G$ into G is jointly continuous.

Definition 2.1. [12] Let X be a subspace of a paratopological group G . Assume that

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- (1) The set X generates G algebraically, that is $\langle X \rangle = G$;
- (2) Each continuous mapping $f : X \rightarrow H$ to a paratopological group H extends to a continuous homomorphism $\hat{f} : G \rightarrow H$.

Then G is called the *Markov free paratopological group on X* and is denoted by $FP(X)$.

Again, if all the groups in the above definitions are Abelian, then we get the definition of the *Markov free Abelian paratopological group on X* which will be denoted by $AP(X)$.

By [12], FPX and $AP(X)$ exist for every space X and the underlying abstract groups of FPX and $AP(X)$ are the free groups on the underlying set of the topological space X respectively. We denote these abstract groups by $FP_a(X)$ and $AP_a(X)$ respectively.

Since X generates the free group $FP_a(X)$, each element $g \in FP_a(X)$ has the form $g = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n}$, where $x_1, \dots, x_n \in X$ and $\varepsilon_1, \dots, \varepsilon_n = \pm 1$. This word for g is called *reduced* if it contains no pair of consecutive symbols of the form xx^{-1} or $x^{-1}x$. It follow that if the word g is reduced and non-empty, then it is different from the neutral element of $FP_a(X)$. For every non-negative integer n , denote by $FP_n(X)$ and $AP_n(X)$ the subspace of paratopological groups $FP(X)$ and $AP(X)$ that consists of all words of reduced length $\leq n$ with respect to the free basis X , respectively.

Let X be a T_1 -space. For each $n \in \mathbb{N}$, denote by i_n the multiplication mapping from $(X \oplus X_d^{-1} \oplus \{e\})^n$ to $B_n(X)$, $i_n(y_1, \dots, y_n) = y_1 \cdots y_n$ for every point $(y_1, \dots, y_n) \in (X \oplus X_d^{-1} \oplus \{e\})^n$, where X_d^{-1} denotes the set X^{-1} equipped with the discrete topology and $B_n(X)$ denotes $FP_n(X)$ or $AP_n(X)$.

By a *quasi-uniform space* (X, \mathcal{U}) we mean the natural analog of a *uniform space* obtained by dropping the symmetry axiom. For each quasi-uniformity \mathcal{U} the filter \mathcal{U}^{-1} consisting of the inverse relations $U^{-1} = \{(y, x) : (x, y) \in U\}$ where $U \in \mathcal{U}$ is called the *conjugate quasi-uniformity* of \mathcal{U} .

Let X be a topological space. Then X_d denotes X when equipped with the discrete topology in place of its given topology. We denote the diagonals of $X_d \times X$ and $X \times X_d$ by Δ_1 and Δ_2 , respectively. In [10], the authors proved that X^{-1} is discrete in free paratopological group $FP(X)$ and $AP(X)$ over X if X is a T_1 -space. We denote the sets $\{(x^{-1}, y) : (x, y) \in X \times X\}$ and $\{(x, y^{-1}) : (x, y) \in X \times X\}$ by Δ_1^* and Δ_2^* , respectively.

3. MAIN RESULTS

Theorem 3.1. [3] *If X is a T_1 -space, then the mapping*

$$i_2 |_{i_2^{-1}(FP_2(X) \setminus FP_1(X))} : i_2^{-1}(FP_2(X) \setminus FP_1(X)) \longrightarrow FP_2(X) \setminus FP_1(X)$$

is a homeomorphism.

Theorem 3.2. [2] *Let X be a T_1 -space and let $w = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}$ be a reduced word in $FP_n(X)$, where $x_i \in X$ and $\varepsilon_i = \pm 1$, for all $i = 1, 2, \dots, n$, and if $x_i = x_{i+1}$ for some $i = 1, 2, \dots, n-1$, then $\varepsilon_i = \varepsilon_{i+1}$. Then the collection \mathcal{B} of all sets of the form $U_1^{\varepsilon_1} U_2^{\varepsilon_2} \cdots U_n^{\varepsilon_n}$, where, for all $i = 1, 2, \dots, n$, the set U_i is a neighborhood of x_i in X when $\varepsilon_i = 1$ and $U_i = \{x_i\}$ when $\varepsilon_i = -1$ is a base for the neighborhood system at w in $FP_n(X)$.*

Theorem 3.3. [2] *Let X be a T_1 -space and let $w = \varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n$ be a reduced word in $AP_n(X)$, where $x_i \in X$ and $\varepsilon_i = \pm 1$, for all $i = 1, 2, \dots, n$, and if $x_i = x_j$ for some $i, j = 1, 2, \dots, n$, then $\varepsilon_i = \varepsilon_j$. Then the collection \mathcal{B} of all sets of the form $\varepsilon_1 U_1 + \varepsilon_2 U_2 + \cdots + \varepsilon_n U_n$, where, for all $i = 1, 2, \dots, n$, the set U_i is a neighborhood of x_i in X when $\varepsilon_i = 1$ and $U_i = \{x_i\}$ when $\varepsilon_i = -1$ is a base for the neighborhood system at w in $AP_n(X)$.*

Theorem 3.4. *If X is a T_1 -space, then the mapping*

$$f = i_2 |_{i_2^{-1}(AP_2(X) \setminus AP_1(X))} : i_2^{-1}(AP_2(X) \setminus AP_1(X)) \longrightarrow AP_2(X) \setminus AP_1(X)$$

is a 2 to 1, open and perfect mapping.

Proof. Obviously, f is a 2 to 1 mapping. Next, we shall prove that f is open and closed. Let $C_2(X) = AP_2(X) \setminus AP_1(X)$ and $C_2^*(X) = i_2^{-1}(AP_2(X) \setminus AP_1(X))$. Obviously, we have

$$C_2^*(X) = (X \times X) \bigoplus (X_d^{-1} \times X_d^{-1}) \bigoplus (X_d^{-1} \times X) \setminus \Delta_1^* \bigoplus (X \times X_d^{-1}) \setminus \Delta_2^*.$$

(1) The mapping f is open.

Let $(x_1^{\epsilon_1}, x_2^{\epsilon_2}) \in C_2^*(X)$, where $x_1, x_2 \in X$ and $x_1 \neq x_2$ if $\epsilon_1 \neq \epsilon_2$. Let U be a neighborhood of $(x_1^{\epsilon_1}, x_2^{\epsilon_2})$ in $C_2^*(X)$. By Theorem 3.3, $f(U)$ is a neighborhood of $x_1^{\epsilon_1} x_2^{\epsilon_2}$ in $C_2(X)$. (Indeed, the argument is similar to the proof of [3, Theorem 3.4].) Therefore, f is open.

(2) The mapping f is closed.

Let E be a closed subset of $C_2^*(X)$. To show that $i_2(E)$ is closed in $C_2(X)$ take $w \in \overline{i_2(E)}$. Next, we shall show that $w \in i_2(E)$. Indeed, it is obvious that w has a reduced form $w = \epsilon_1 x_1 + \epsilon_2 x_2$, where $\epsilon_i = 1$ or -1 ($i = 1, 2$), $x_1, x_2 \in X$ and $x_1 \neq x_2$ if $\epsilon_1 \neq \epsilon_2$.

Suppose that $w = x + y \notin i_2(E)$, where $x = \epsilon_1 x_1$ and $y = \epsilon_2 x_2$. Then $\{(x, y), (y, x)\} \cap E = \emptyset$. Since E is closed, we can pick open neighborhoods $V(x)$ of x in $X \cup X_d^{-1}$, $V(y)$ of y in $X \cup X_d^{-1}$ such that $(V(x) \times V(y)) \cap E = \emptyset$, $(V(y) \times V(x)) \cap E = \emptyset$. Let $U = (V(x) \times V(y)) \cup (V(y) \times V(x))$. Then U is open. Since f is an open map, we have $f(U)$ is a neighborhood of w and $f(U) \cap i_2(E) = \emptyset$. This contradicts with $w \in \overline{i_2(E)}$. \square

For arbitrary space X , the mapping $f : X \rightarrow \mathbb{Z}$ defined by setting $f(x) = 1$ for all $x \in X$ is continuous, and thus extends to a continuous homomorphism $\hat{f} : AP(X) \rightarrow \mathbb{Z}$. Therefore, the collection of sets $Z_n(X) = \hat{f}^{-1}(\{n\})$ for $n \in \mathbb{Z}$ forms a partition of $AP(X)$ into clopen subspaces.

For a T_1 -space, define

$$g : (X_d \times X) \bigoplus (X \times X_d) \bigoplus (\{e\} \times \{e\}) \rightarrow AP_2(X) \cap Z_0(X)$$

by

$$g(x, y) = \begin{cases} -x + y, & \text{if } (x, y) \in X_d \times X; \\ x - y, & \text{if } (x, y) \in X \times X_d; \\ e, & \text{if } x = y. \end{cases}$$

Let $g_j = i_2|_{i_2^{-1}(AP_2(X) \cap Z_j(X))}$ for $j = -2, \dots, 2$, where $i_2 : (X \bigoplus X_d^{-1} \bigoplus \{e\})^2 \rightarrow AP_2(X)$. Obviously, $i_2 = \bigoplus_{j=-2}^{j=2} \{g_j\}$, and i_2 is a closed (resp. quotient) mapping if and only if each g_j is a closed (resp. quotient) mapping, where $j = -2, \dots, 2$. By Theorem 3.4, it is easy to see that g_{-2} and g_2 are open and closed. Moreover, since $-X$ occurs with the discrete topology and X occurs with its original topology in $AP(X)$, the mappings g_{-1} and g_1 are open and closed. Obviously, g is a closed (resp. quotient) mapping if and only if g_0 is a closed (resp. quotient) mapping. Therefore, we have the following result:

Lemma 3.5. *Let X be a T_1 -space. Then i_2 is a closed (resp. quotient) mapping if and only if g is a closed (resp. quotient) mapping.*

Lemma 3.6. [3] *Let X be a space and let Δ_1 be the diagonal in the space $X_d \times X$. Then Δ_1 is closed if and only if X is T_1 . Similarly for the diagonal Δ_2 in the space $X \times X_d$.*

Suppose that \mathcal{U}^* is the finest quasi-uniformity of a space X . We say that $P = \{U_i\}_{i \in \mathbb{N}}$ is a sequence of \mathcal{U}^* if each $U_i \in \mathcal{U}^*$. Put

$${}^\omega \mathcal{U}^* = \{P : P \text{ is a sequence of } \mathcal{U}^*\}.$$

For each $n \in \mathbb{N}$ and $P = \{U_i\}_{i \in \mathbb{N}} \in {}^\omega \mathcal{U}^*$, let

$$\mathcal{Q}_n(\mathbb{N}) = \{A \subset \mathbb{N} : |A| = n\},$$

$$W_n(P) = \{-x_1 + y_1 - \dots - x_n + y_n : (x_j, y_j) \in U_{i_j} \text{ for } j = 1, 2, \dots, n, \{i_1, i_2, \dots, i_n\} \in \mathcal{Q}_n(\mathbb{N})\},$$

and

$$\mathcal{W}_n = \{W_n(P) : P \in {}^\omega \mathcal{U}^*\}.$$

Remark In the above definition, for $P = \{U_i\}_{i \in \mathbb{N}} \in {}^\omega \mathcal{U}^*$, there may exist $i \neq j$ such that $U_i = U_j$. In particular, for every $U \in \mathcal{U}^*$, we have $\{U_i = U\}_{i \in \mathbb{N}}$ is also in ${}^\omega \mathcal{U}^*$. Moreover,

the reader should note that the representation of elements of $W_n(P)$ need not be a reduced representation.

Theorem 3.7. [7] *For each $n \in \mathbb{N}$, the family \mathcal{W}_n is a neighborhood base of e in $AP_{2n}(X)$.*

The proof of the following Theorem is a modification of [3, Theorem 3.10].

Theorem 3.8. *Let X be a T_1 -space. Then the mapping*

$$i_2 : (X \oplus X_d^{-1} \oplus \{e\})^2 \longrightarrow AP_2(X)$$

is a quotient mapping if and only if every neighborhood U of the diagonal Δ_1 in $X_d \times X$ is a member of the finest quasi-uniformity \mathcal{U}^ on X .*

Proof. Put $Z = (X_d \times X) \oplus (X \times X_d) \oplus (\{e\} \times \{e\})$.

Necessity. Suppose that i_2 is a quotient mapping. It follows from Lemma 3.5 that $g : Z \longrightarrow AP_2(X) \cap Z_0(X)$ is a quotient mapping. Let U be a neighborhood of Δ_1 in $X_d \times X$. Obviously, $U \cup (-U)$ is a neighborhood of $\Delta_1 \cup \Delta_2$ in Z . Let $P = \{U_n\}_{n \in \mathbb{N}}$, where $U_n = U$ for each $n \in \mathbb{N}$. Let $W_1(P) = \{-x + y : (x, y) \in U\}$. Then $g^{-1}(W_1(P)) = U \cup (-U) \cup \{(e, e)\}$ that is a neighborhood of $\Delta_1 \cup \Delta_2 \cup \{(e, e)\}$ in Z , then $W_1(P)$ is a neighborhood of e in $AP_2(X) \cap Z_0(X)$, and hence in $AP_2(X)$. By Theorem 3.7, there exists $Q \in {}^\omega \mathcal{U}^*$ such that $W_1(Q) \subset W_1(P)$, where $Q = \{V_n\}_{n \in \mathbb{N}}$. Then $V_1 \subset U$, hence $U \in \mathcal{U}^*$.

Sufficiency. Suppose that every neighborhood U of the diagonal Δ_1 in $X_d \times X$ is a member of the finest quasi-uniformity \mathcal{U}^* on X . To show that i_2 is a quotient mapping, it follows from Lemma 3.5 that it suffices to show that the mapping $g : Z \longrightarrow AP_2(X) \cap Z_0(X)$ is a quotient mapping. Take a subset $A \subset AP_2(X) \cap Z_0(X)$ such that $g^{-1}(A)$ is open in Z . Put $U = g^{-1}(A) \cap (X_d \times X)$ and $V = g^{-1}(A) \cap (X \times X_d)$. Firstly, we show the following claim:

Claim: If $e \notin A$, then A is open in $AP_2(X) \cap Z_0(X)$.

Indeed, since $e \notin A$, $U \cap \Delta_1 = \emptyset$ and $V \cap \Delta_2 = \emptyset$. By Lemma 3.6, Δ_1 and Δ_2 are closed in $X_d \times X$ and $X \times X_d$, respectively, and $X_d \times X \setminus \Delta_1$ and $X \times X_d \setminus \Delta_2$ are open in $X_d \times X$ and $X \times X_d$, respectively. Hence $U \cup V$ is open in the space $i_2^{-1}(AP_2(X) \setminus AP_1(X))$, and by Theorem 3.4, $g(U \cup V) = A$ is open in $AP_2(X) \cap Z_0(X)$.

Next we shall show that A is open in $AP_2(X) \cap Z_0(X)$. Take arbitrary $a \in A$. Then it suffices to show that A is open neighborhood of a .

Case 1: $a = e$.

Obviously, U and V are open neighborhoods of Δ_1 and Δ_2 in $X_d \times X$ and $X \times X_d$, respectively. Therefore, $S = U \cap (V^{-1})$ is an open neighborhood of Δ_1 in $X_d \times X$, and thus $S \in \mathcal{U}^*$. Let $W_1(R) = \{-x + y : (x, y) \in S\}$, where $R = \{S_n\}_{n \in \mathbb{N}}$ and $S_n = S$ for each $n \in \mathbb{N}$. By Theorem 3.7, $W_1(R)$ is a neighborhood of e in $AP_2(X)$. Since $S = U \cap (V^{-1})$ and the definition of g , it is easy to see that $W_1(R) \subset A$. Therefore, A is a neighborhood of e in $AP_2(X)$, hence in $AP_2(X) \cap Z_0(X)$.

Case 2: $a \neq e$.

Let W be an open neighborhood of a in $AP_2(X) \cap Z_0(X)$ such that $e \notin W$. Then the set $g^{-1}(A \cap W)$ is open in Z , and it follows from Claim that $A \cap W$ is an open neighborhood of a in $AP_2(X) \cap Z_0(X)$. Hence A is open in $AP_2(X) \cap Z_0(X)$. \square

The next theorem is the main result in [3], and some related concepts can be seen in [5]. Next, we shall improve this result in Theorem 3.11.

Theorem 3.9. [3] *Let X be a T_1 -space. Then the followings are equivalent:*

- (1) *The mapping $i_2 : (X \oplus X_d^{-1} \oplus \{e\})^2 \longrightarrow FP_2(X)$ is a quotient mapping;*
- (2) *Every neighborhood U of the diagonal Δ_1 in $X_d \times X$ is a member of the finest quasi-uniformity \mathcal{U}^* on X ;*
- (3) *Every neighbornet of X is normal;*
- (4) *The finest quasi-uniformity \mathcal{U}^* on X consists of all neighborhoods of the diagonal Δ_1 in $X_d \times X$;*

- (5) If N_x is a neighborhood of x for all $x \in X$, then there exists a neighborhood M_x of x such that $\bigcup_{y \in M_x} M_y \subset N_x$ for all $x \in X$;
- (6) If N_x is a neighborhood of x for all $x \in X$, then there exists a quasi-pseudometric d on X such that d_x is upper semi-continuous and $B_d(x, 1) \subset N_x$ for all $x \in X$.

Let X be a set. Define $j_2, k_2 : X \times X \longrightarrow F_a(X)$ by $j_2(x, y) = x^{-1}y$ and $k_2(x, y) = yx^{-1}$.

Theorem 3.10. [3] *Let X be a topological space. Then the collection \mathcal{B} of sets $j_2(U) \cup k_2(U)$ for $U \in \mathcal{U}^*$ is a base of neighborhoods at the identity e in $FP_2(X)$.*

Now we can prove the main theorem in this paper.

Theorem 3.11. *Let X be a T_1 -space. Then the following are equivalent:*

- (1) *The mapping $i_2 : (X \oplus X_d^{-1} \oplus \{e\})^2 \longrightarrow FP_2(X)$ is a quotient mapping;*
- (2) *The mapping $i_2 : (X \oplus X_d^{-1} \oplus \{e\})^2 \longrightarrow AP_2(X)$ is a quotient mapping;*
- (3) *The mapping $i_2 : (X \oplus X_d^{-1} \oplus \{e\})^2 \longrightarrow FP_2(X)$ is a closed mapping;*
- (4) *The mapping $i_2 : (X \oplus X_d^{-1} \oplus \{e\})^2 \longrightarrow AP_2(X)$ is a closed mapping.*

Proof. Obviously, we have (3) \Rightarrow (1) and (4) \Rightarrow (2). Moreover, it follows from Theorems 3.8 and 3.9 that we have (2) \Rightarrow (1). It suffices to show that (1) \Rightarrow (3) and (2) \Rightarrow (4).

(1) \Rightarrow (3). Clearly, both $FP_2(X) \setminus FP_1(X)$ and $FP_1(X) \setminus \{e\}$ are open in $FP_2(X)$. Let E be a closed subset in $(X \oplus X_d^{-1} \oplus \{e\})^2$. To show that $i_2(E)$ is closed in $FP_2(X)$ take $w \in \overline{i_2(E)}$.

Case a1: $w \in FP_1(X) \setminus \{e\}$.

Suppose $w \notin i_2(E)$, then $(w, e) \notin E$ and $(e, w) \notin E$. Since E is closed, there is open neighborhood U (open in $X \cup X_d^{-1}$) of w such that $(U \times \{e\}) \cap E = \emptyset$ and $(\{e\} \times U) \cap E = \emptyset$. Obviously, we have $(U \times \{e\}) \cup (\{e\} \times U) = i_2^{-1}(U)$. Then U is open in $FP_2(X)$ since $(U \times \{e\}) \cup (\{e\} \times U)$ is open in $(X \oplus X_d^{-1} \oplus \{e\})^2$ and i_2 is a quotient map. Hence $U \cap i_2(E) = \emptyset$, which contradicts $w \in \overline{i_2(E)}$.

Case a2: $w \in FP_2(X) \setminus FP_1(X)$.

Let $w = w_1^{\epsilon_1} w_2^{\epsilon_2}$, where $w_i \in X$ and $\epsilon_i = 1$ or -1 ($i = 1, 2$). Suppose that $w \notin i_2(E)$. Then $(w_1^{\epsilon_1}, w_2^{\epsilon_2}) \notin E$.

Subcase a21: $\epsilon_1 = \epsilon_2 = 1$.

Since $(w_1, w_2) \notin E$ and E is closed in $(X \oplus X_d^{-1} \oplus \{e\})^2$, there exist neighborhoods U and V of w_1 and w_2 in X , respectively, such that $(U \times V) \cap E = \emptyset$. Therefore, it is easy to see that $UV \cap i_2(E) = \emptyset$. From Theorem 3.2 it follows that UV is a neighborhood of w , hence $w \notin \overline{i_2(E)}$, which is a contradiction.

Subcase a22: $\epsilon_1 = \epsilon_2 = -1$.

From Theorem 3.2 it follows that $\{w_1^{-1} w_2^{-1}\}$ is a neighborhood of w , then $w \notin \overline{i_2(E)}$, which is a contradiction.

Subcase a23: $\epsilon_1 \neq \epsilon_2$.

Without loss of generality, we may assume that $\epsilon_1 = 1$ and $\epsilon_2 = -1$. Then since $(w_1, w_2^{-1}) \notin E$ and E is closed in $(X \oplus X_d^{-1} \oplus \{e\})^2$, there exists a neighborhood U of w_1 in X such that $(U \times \{w_2^{-1}\}) \cap E = \emptyset$ and $w_2 \notin U$. (This is possible since X is T_1 .) Obviously, $U w_2^{-1} \subset FP_2(X) \setminus FP_1(X)$. Therefore, it is easy to see that $\overline{U w_2^{-1}} \cap i_2(E) = \emptyset$. From Theorem 3.2 it follows that $U w_2^{-1}$ is a neighborhood of w , hence $w \notin \overline{i_2(E)}$, which is a contradiction.

Therefore, we have $w \in i_2(E)$.

Case a3: $w = e$.

Suppose that $e \notin i_2(E)$. Then $E \cap (\Delta_1 \cup \Delta_2 \cup \{(e, e)\}) = \emptyset$. For any $x \in X$, since E does not contain points (x^{-1}, x) and (x, x^{-1}) , there exists an open neighborhood $U(x)$ of x in X such that $(\{x^{-1}\} \times U(x)) \cap E = \emptyset$ and $(U(x) \times \{x^{-1}\}) \cap E = \emptyset$. Let $U = \bigcup_{x \in X} (\{x^{-1}\} \times U(x))$ and $V = \bigcup_{x \in X} (U(x) \times \{x^{-1}\})$. Then $U \cap E = \emptyset$ and $V \cap E = \emptyset$. Let $W = U \cup V \cup \{e\} \times \{e\}$. Then W is open in $(X \oplus X_d^{-1} \oplus \{e\})^2$ by (2) of Theorem 3.9. Obviously, we have $W \cap E = \emptyset$. It is easy to see that $i_2^{-1}(i_2(W)) = W$, then $i_2(W)$ is open since i_2 is a quotient map. Hence $i_2(W) \cap i_2(E) = \emptyset$, this is a contradiction.

(2) \Rightarrow (4). (**Note:** The proof is almost similar to (1) \Rightarrow (3). However, we give out the proof for the convenience for readers.) Clearly, both $AP_2(X) \setminus AP_1(X)$ and $AP_1(X) \setminus \{e\}$ are open in $AP_2(X)$. Let E be a closed subset in $(X \oplus -X_d \oplus \{e\})^2$. To show that $i_2(E)$ is closed in $AP_2(X)$ take $w \in \overline{i_2(E)}$.

Case b1: $w \in AP_1(X) \setminus \{e\}$.

Suppose $w \notin i_2(E)$, then $(w, e) \notin E$ and $(e, w) \notin E$. Since E is closed, there is open neighborhood U (open in $X \cup -X_d$) of w such that $(U \times \{e\}) \cap E = \emptyset$ and $(\{e\} \times U) \cap E = \emptyset$. Obviously, we have $(U \times \{e\}) \cup (\{e\} \times U) = i_2^{-1}(U)$. Then U is open in $AP_2(X)$ since $(U \times \{e\}) \cup (\{e\} \times U)$ is open in $(X \oplus -X_d \oplus \{e\})^2$ and i_2 is a quotient map by Theorems 3.8 and 3.9. Then $U \cap i_2(E) = \emptyset$, that contradicts $w \in \overline{i_2(E)}$.

Case b2: $w \in AP_2(X) \setminus AP_1(X)$.

Let $w = \epsilon_1 w_1 + \epsilon_2 w_2$, where $w_i \in X$ and $\epsilon_i = 1$ or -1 ($i = 1, 2$). Suppose that $w \notin i_2(E)$. Then $(\epsilon_1 w_1, \epsilon_2 w_2) \notin E$ and $(\epsilon_2 w_2, \epsilon_1 w_1) \notin E$.

Subcase b21: $\epsilon_1 = \epsilon_2 = 1$.

Since $\{(w_1, w_2), (w_2, w_1)\} \notin E$ and E is closed in $(X \oplus -X_d \oplus \{e\})^2$, there exist neighborhoods U and V of w_1 and w_2 in X , respectively, such that $(U \times V \cup V \times U) \cap E = \emptyset$. Therefore, it is easy to see that $(U + V) \cap i_2(E) = \emptyset$. From Theorem 3.3 it follows that $U + V$ is a neighborhood of w , hence $w \notin \overline{i_2(E)}$, which is a contradiction.

Subcase b22: $\epsilon_1 = \epsilon_2 = -1$.

From Theorem 3.2 it follows that $\{-w_1 - w_2\}$ is a neighborhood of w , then $w \notin \overline{i_2(E)}$, which is a contradiction.

Subcase b23: $\epsilon_1 \neq \epsilon_2$.

Without loss of generality, we may assume that $\epsilon_1 = 1$ and $\epsilon_2 = -1$. Then since

$$\{(w_1, -w_2), (-w_2, w_1)\} \notin E$$

and E is closed in $(X \oplus -X_d \oplus \{e\})^2$, there exists a neighborhood U of w_1 in X such that $(U \times \{w_2^{-1}\} \cup \{w_2^{-1}\} \times U) \cap E = \emptyset$ and $w_2 \notin U$. (This is possible since X is T_1 .) Obviously, $U - w_2 \subset AP_2(X) \setminus AP_1(X)$. Therefore, it is easy to see that $(U - w_2) \cap i_2(E) = \emptyset$. From Theorem 3.3 it follows that $U - w_2$ is a neighborhood of w , hence $w \notin \overline{i_2(E)}$, which is a contradiction.

Therefore, we have $w \in i_2(E)$.

Case b3: $w = e$.

Suppose that $e \notin i_2(E)$. Then $E \cap (\Delta_1 \cup \Delta_2 \cup \{(e, e)\}) = \emptyset$. For any $x \in X$, since E does not contain points $(-x, x)$ and $(x, -x)$, there exists an open neighborhood $U(x)$ of x in X such that $(\{-x\} \times U(x)) \cap E = \emptyset$ and $(U(x) \times \{-x\}) \cap E = \emptyset$. Let $U = \bigcup_{x \in X} (\{-x\} \times U(x))$ and $V = \bigcup_{x \in X} (U(x) \times \{-x\})$. Then $U \cap E = \emptyset$ and $V \cap E = \emptyset$. Let $W = U \cup V \cup \{e\} \times \{e\}$. Then W is open in $(X \oplus -X_d \oplus \{e\})^2$ by Theorem 3.9. Obviously, we have $W \cap E = \emptyset$. It is easy to see that $i_2^{-1}(i_2(W)) = W$, then $i_2(W)$ is open in $AP_2(X)$ since i_2 is a quotient map by Theorems 3.8 and 3.9. Hence $i_2(W) \cap i_2(E) = \emptyset$, which is a contradiction. \square

Proposition 3.12. Let X be a T_1 -space. Then, for some $n \geq 3$, $i_n : (X \oplus X_d^{-1} \oplus \{e\})^n \rightarrow FP_n(X)$ is a closed map if and only if X is discrete.

Proof. If X is discrete, then $FP(X)$ is discrete, it is easy to see that each i_n is a closed map.

Let i_n be a closed map for some $n \geq 3$. Since X is T_1 , then X^{-1} is discrete. Suppose that X is not discrete, then there exists $x \in X$ such that $x \in \overline{X \setminus \{x\}}$. Let

$$A = \{(x_\alpha, x_\alpha, x_\alpha^{-1}, e, \dots, e) \in (X \oplus X_d^{-1} \oplus \{e\})^n : x_\alpha \in X \setminus \{x\}\}.$$

Then A is a closed discrete subset of $(X \oplus X_d^{-1} \oplus \{e\})^n$, and therefore, $i_n(A) = X \setminus \{x\}$ is closed discrete subset, which is a contradiction. Hence X is discrete. \square

Note Therefore, we can improve all results in [3, Sections 4 and 5] from quotient mappings to closed mappings. For example, we have the following proposition.

Proposition 3.13. The mapping i_2 is a closed mapping for any countable T_1 -space. In particular, the mapping i_2 is a closed mapping for any countable subspace of real line \mathbb{R} .

Corollary 3.14. $FP_2(\mathbb{Q})$ and $AP_2(\mathbb{Q})$ are Fréchet, where \mathbb{Q} is the rational number of real line \mathbb{R} .

Proof. By Proposition 3.13, i_2 is a closed mapping. Then $FP_2(\mathbb{Q})$ and $AP_2(\mathbb{Q})$ are Fréchet since $(X \oplus X_d^{-1} \oplus \{e\})^2$ is Fréchet and closed mappings preserve the property of Fréchet. \square

By [5, Proposition 6.26], we also have the following proposition.

Proposition 3.15. For arbitrary compact first-countable Hausdorff space X , the mapping i_2 is closed if and only if X is countable.

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